Unit 6, Lesson 1: Tape Diagrams and Equations

Lesson Goals

- Use tape diagrams to reason about unknown values in equations of the form $x + p = q$ and $px = q$.
- Use tape diagrams to reason about writing the equations $x + p = q$ and $px = q$ in different forms.
- Represent equations of the form $x + p = q$ and $px = q$ with tape diagrams.

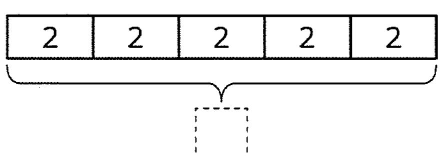
Required Materials

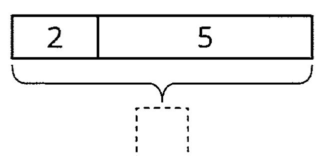
1.1: Which Diagram is Which? (5 minutes)

Setup: 2 minutes of quiet think time, followed by a whole-class discussion.

Student task statement

Here are two diagrams. One represents $2 + 5 = 7$. The other represents $5 \cdot 2 = 10$. Which is which? Label the length of each diagram.



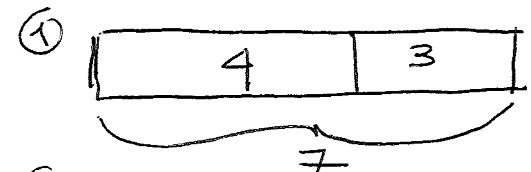


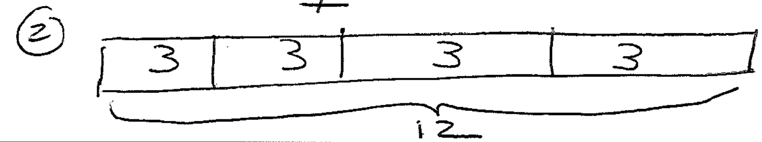
Draw a diagram that represents each equation.

1. $4 + 3 = 7$
2. $4 \cdot 3 = 12$

Possible responses

$5 \cdot 2 = 10$ on the left and $2 + 5 = 7$ on the right. For the other diagrams, see lesson plan. ↓



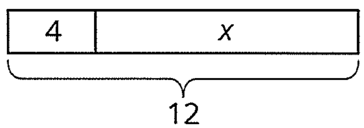


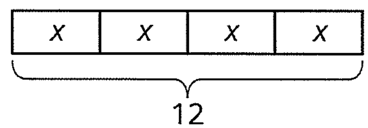
1.2: Match Equations and Tape Diagrams (10 minutes)

Setup: 5 minutes of quiet think time, followed by a whole-class discussion.

Student task statement

Here are two tape diagrams. Match each equation to one of the tape diagrams.





1. $4 + x = 12$
2. $12 \div 4 = x$
3. $4 \cdot x = 12$
4. $12 = 4 + x$
5. $12 - x = 4$
6. $12 = 4 \cdot x$
7. $12 - 4 = x$
8. $x = 12 - 4$
9. $x + x + x + x = 12$

Possible responses

Left diagram: 1, 4, 5, 7, 8. Right diagram: 2, 3, 6, 9.

Anticipated misconceptions

Students may not have much experience with a letter standing in for a number. If students resist engaging, explain that the x is just standing in for a number. Students may prefer to figure out the value that x must take to make each diagram make sense (8 in the first diagram and 3 in the second diagram) before thinking out which equations can represent each diagram.

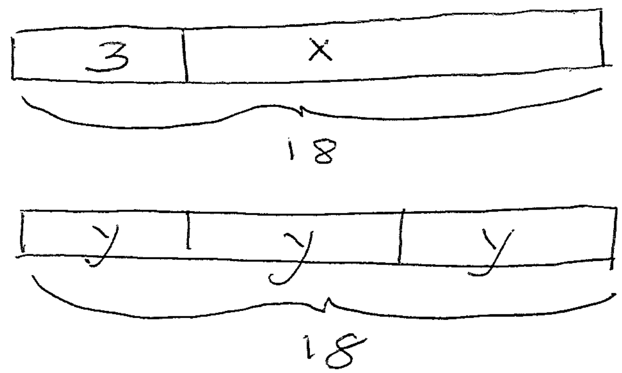
1.3: Draw Diagrams for Equations (15 minutes)

Setup: 5 minutes of quiet work time, followed by a whole-class discussion.

Student task statement

For each equation, draw a diagram and find the value of the unknown that makes the equation true.

1. $18 = 3 + x$
2. $18 = 3 \cdot y$



Possible responses

1. 15
2. 6

see lesson plan for diagrams

Anticipated misconceptions

Students might draw a box with 3 for the equation $18 = 3 \cdot y$. Ask students about the meaning of multiplication and specifically what $3 \cdot y$ means. Ask how they could represent 3 equal groups with unknown size y.

Students might think they need to show an unknown number (y) of equal groups of 3. While this is possible, showing 3 equal groups with unknown size y is simpler to represent.

Are you ready for more?

You are walking down a road, seeking treasure. The road branches off into three paths. A guard stands in each path. You know that only one of the guards is telling the truth, and the other two are lying. Here is what they say:

- Guard 1: The treasure lies down this path.
- Guard 2: No treasure lies down this path; seek elsewhere.
- Guard 3: The first guard is lying.

Which path leads to the treasure?

Possible Responses

Path 2 leads to the treasure.

Suppose Guard 1 is telling the truth. Then it would be true that Path 1 leads to the treasure. Then Guard 2's statement must be true as well. But only one of the guards is telling the truth. This means that Guard 1 must be lying.

Since Guard 1 is lying, Guard 3 is telling the truth about Guard 1 lying. That means that Guard 3 has the one true statement. Guard 2, then, is lying about his path being the wrong path, so the treasure lies down Path 2.

Lesson Synthesis (5 minutes)

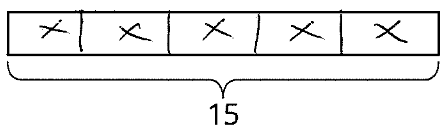
How do tape diagrams help us reason about relationships between quantities? Can a tape diagram and an equation represent the same relationship? Ensure students understand that tape diagrams can help us visualize relationships and find unknown amounts, and that tape diagrams and equations can both represent relationships between quantities.

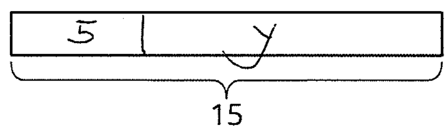
1.4: Finish the Diagrams (Cool-down, 5 minutes)

Setup: None.

Student task statement

Finish the first diagram so that it represents $5 \cdot x = 15$, and the second diagram so that it represents $5 + y = 15$.





Possible responses

See lesson plan.

Unit 6, Lesson 2: Truth and Equations

Lesson Goals

- Represent problems in context with equations of the form $x + p = q$ and $px = q$.
- Understand that an equation can be true or false.
- Understand that, in an equation with a variable, a value substituted for the variable that makes the equation true is called a solution to the equation.
- Use substitution to determine whether a given number in a specified set makes an equation true.
- Understand that a letter standing in for a number is called a variable.

Required Materials

2.1: Three Letters (10 minutes)

Setup:

2 minutes of quiet work time for the first part of the first question. Explain the meaning of variable and demonstrate why $a + b = c$ is false for 3, 4 and 5. 2 minutes to complete the rest of the task, followed by a whole-class discussion.

Student task statement

1. The equation $a + b = c$ could be true or false.

   a. If a is 3, b is 4, and c is 5, is the equation true or false?

   b. Find new values of a, b, and c that make the equation true.

   c. Find new values of a, b, and c that make the equation false.

2. The equation $x \cdot y = z$ could be true or false.

   a. If x is 3, y is 4, and z is 12, is the equation true or false?

   b. Find new values of x, y, and z that make the equation true.

   c. Find new values of x, y, and z that make the equation false.

Possible responses

Answers vary. Sample responses:

1. a. False

   b. a is 1, b is 2, c is 3

   c. a is 4, b is 5, c is 10

2. a. True

   b. x is 3, y is 5, z is 15

   c. x is 1, y is 2, z is 3

2.2: Storytime (15 minutes)

Setup:

Introduce the "next to" notation. Explain that $20x$ means the same thing as $20 \cdot x$, and that the 20 in $20x$ is called the "coefficient." 5 minutes of quiet work time, followed by a whole-class discussion.

Student task statement

Here are three situations and six equations. Which equation best represents each situation? If you get stuck, draw a diagram.

1. After Elena ran 5 miles on Friday, she had run a total of 20 miles for the week. She ran x miles before Friday.
2. Andre's school has 20 clubs, which is five times as many as his cousin's school. His cousin's school has x clubs.
3. Jada volunteers at the animal shelter. She divided 5 cups of cat food equally to feed 20 cats. Each cat received x cups of food.

$x+5=20$	$x=20+5$	$5x=20$
$x+20=5$	$5\cdot 20=x$	$20x=5$

Possible responses

1. $x+5=20$
2. $5x=20$
3. $20x=5$

Anticipated misconceptions

Students who focus on key words might be misled in each situation. For the first situation, students might see the word "total" and decide they need to add 5 and 20. In the second situation, the words "five times as many" might prompt them to multiply 5 by 20. The third story poses some additional challenges: students see the word "divided" but there is no equation with division. Additionally, students might think that division always means divide the larger number by the smaller. Here are some ways to help students make sense of the situations and how equations can represent them:

- Suggest that students act out the situation or draw a picture. Focus on what quantity in the story each number or variable represents, and on the relationships among them.
- Use tape diagrams to represent quantities and think about where a situation describes a total and where it describes parts of the total.
- Ask students about the relationships between operations. For the third situation, ask what operation is related to dividing and might help describe the situation.

2.3: Using Structure to Find Solutions (15 minutes)

Setup: Groups of 2. 10 minutes to work quietly and then share responses with a partner, followed by a whole-class discussion.

Student task statement

Here are some equations that contain a **variable** and a list of values. Think about what each equation means and find a **solution** in the list of values. If you get stuck, draw a diagram. Be prepared to explain why your solution is correct.

1. $1000 - a = 400$
2. $12.6 = b + 4.1$
3. $8c = 8$
4. $\frac{2}{3} \cdot d = \frac{10}{9}$
5. $10e = 1$
6. $10 = 0.5f$
7. $0.99 = 1 - g$
8. $h + \frac{3}{7} = 1$

List:

$\frac{1}{8}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{3}$	$\frac{7}{3}$	0.01	0.1	0.5
1	2	8.5	9.5	16.7	20	400	600	1400

Possible responses

1. 600
2. 8.5
3. 1
4. $\frac{5}{3}$
5. 0.1
6. 20
7. 0.01
8. $\frac{4}{7}$

Anticipated misconceptions

Instead of solving, students might follow the operation symbol and combine the numbers in that way (for example, adding 12.6 and 4.1 to get 16.7 for the equation $12.6 = b + 4.1$). Encourage students to express the relationships of the equation in words and to draw diagrams that describe those statement. $12.6 = b + 4.1$ can be stated as "When a number is added to 4.1, the sum is 12.6." The tape diagram then shows the parts 4.1 and an unknown quantity b, and a total of 12.6.

Are you ready for more?

One solution to the equation $a + b + c = 10$ is $a = 2, b = 5, c = 3$.

How many different whole-number solutions are there to the equation $a + b + c = 10$? Explain or show your reasoning.

Possible Responses

If $a, b,$ and c are *positive*, there are 36 solutions. If $a = 1$, the possibilities are that $b = 1$ and $c = 8$, $b = 2$ and $c = 7$, and so on, giving 8 solutions. If $a = 2$, then b and c could respectively be 1 and 7, 2 and 6, 3 and 5, etc. This gives 7 solutions for $a = 2$. If $a = 9$ and all numbers are positive, there are no possible numbers for both b and c. The total number of solutions (for a value of 1 through 8) is
$8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$ or 36.

If $a, b,$ and c *non-negative* and includes 0, there are 66 solutions. If $a = 0$, there are 11 combinations of b and c. If $a = 1$, there are 10 combinations, and so on. The total number of solutions (for a value of 0 through 10) is
$11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$, which is 66.

If $a, b,$ and c are *any* integers (including negative), then there is an unlimited number of solutions.

Lesson Synthesis (5 minutes)

Review understanding and appropriate use of new vocabulary (variable, coefficient, solution) and the concepts they represent. What does it mean for an equation to be true? False? Is an equation with a variable always true?

2.4: How Do You Know a Solution is a Solution? (Cool-down, 5 minutes)

Setup: None.

Student task statement

Explain how you know that 88 is a solution to the equation $\frac{1}{8}x = 11$ by completing the sentences:

The word "solution" means . . .

88 is a solution to $\frac{1}{8}x = 11$ because . . .

Possible responses

The word "solution" means a value that makes the equation true.

88 is a solution to $\frac{1}{8}x = 11$, because if x is 88, the equation is $\frac{1}{8} \cdot 88 = 11$, which is true.

Unit 6, Lesson 3: Staying in Balance

Lesson Goals

- Understand how a balanced hanger can represent a true equation.
- Write equations of the form $x + p = q$ and $px = q$ to represent balanced hangers.
- Use balanced hangers to reason about finding solutions to equations of the form $x + p = q$ and $px = q$.

Required Materials

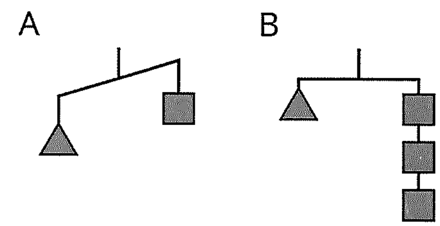
3.1: Hanging Around (10 minutes)

Setup:

Display photo of socks for all to see. 1 minute to notice and wonder. 3 minutes of quiet work time, followed by a whole-class discussion.

Student task statement

1. For diagram A, find:
   a. One thing that *must* be true
   b. One thing that *could* be true or false
   c. One thing that *cannot possibly* be true
2. For diagram B, find:
   a. One thing that *must* be true
   b. One thing that *could* be true or false
   c. One thing that *cannot possibly* be true



Possible responses

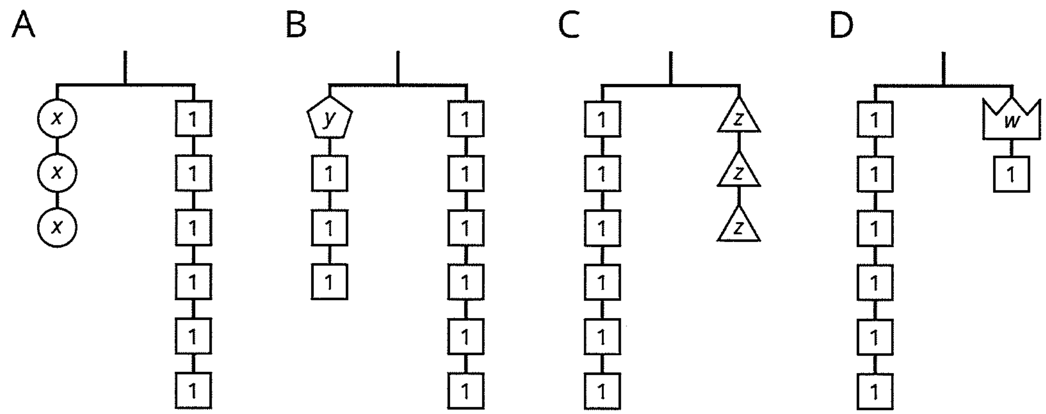
Answers vary. Sample responses:

1. Diagram A:
   a. The triangle is heavier than the square.
   b. The triangle could weigh 10 ounces and the square could weigh 6 ounces.
   c. The square and the triangle weigh the same.
2. Diagram B:
   a. One triangle weighs the same as three squares.
   b. The triangle weighs three pounds and each square weighs one pound.
   c. One square is heavier than the triangle.

3.2: Match Equations and Hangers (15 minutes)

Setup: Groups of 2. 5–10 minutes to work quietly and then share responses with a partner, followed by a whole-class discussion.

Student task statement



1. Match each hanger to an equation. Complete the equation by writing x, y, z, or w in the empty box.

   $\square + 3 = 6$ $\quad 3 \cdot \square = 6 \quad 6 = \square + 1 \quad 6 = 3 \cdot \square$

2. Find a solution to each equation. Use the hangers to explain what each solution means.

Possible responses

1. A: $3 \cdot x = 6$
   B: $y + 3 = 6$
   C: $6 = 3 \cdot z$
   D: $6 = w + 1$

2. A: x is 2, each circle weighs the same as 2 squares.

   B: y is 3, the pentagon weighs as much as 3 squares.

   C: z is 2, the z shape weighs the same as 2 squares.

   D: w is 5, the w shape weighs as much as 5 squares.

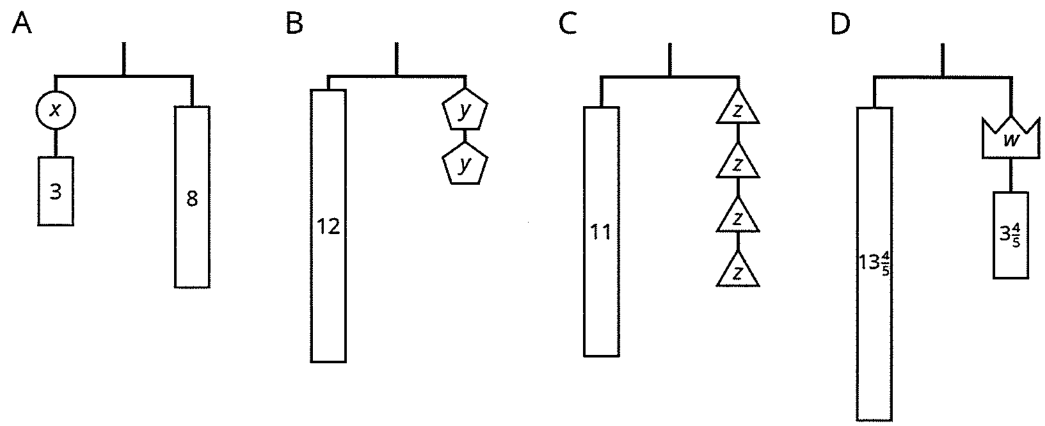
3.3: Connecting Diagrams to Equations and Solutions (15 minutes)

Setup:

Keep students in the same groups. 5–10 minutes to work quietly and then share responses with a partner, followed by a whole-class discussion.

Student task statement

Here are some balanced hangers. Each piece is labeled with its weight.



For each diagram:

1. Write an equation.
2. Explain how to reason with the diagram to find the weight of a piece with a letter.
3. Explain how to reason with the equation to find the weight of a piece with a letter.

Possible responses

1. A: $x + 3 = 8$
   B: $12 = 2y$
   C: $11 = 4z$
   D: $13\frac{4}{5} = w + 3\frac{4}{5}$

2. a. $x = 5$. Together x and 3 weigh 8, so x weighs 5.

   b. $y = 6$. 12 is twice the weight of y, so y weighs half of 12.

   c. $z = \frac{11}{4}$. 11 is 4 times the weight of z, so z weighs a quarter of 11.

   d. $w = 10$. Together w and $3\frac{4}{5}$ weight $13\frac{4}{5}$, so w weighs 10.

3. a. Subtracting 3 from each side of the equation leaves $x = 5$.

   b. The right side of the equation is equal to $2y$. After dividing each side of the equation by 2, the equation is $6 = y$.

c. Dividing each side of the equation by 4 leaves $\frac{11}{4} = z$.

d. Subtracting $3\frac{4}{5}$ from each side of the equation leaves $10 = w$.

Lesson Synthesis (5 minutes)

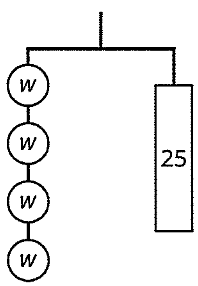
Display the two equations $5x = 8$ and $5 + x = 8$. Ask students to draw a hanger to match each equation. Then have them work with a partner to solve the equation alongside finding the unknown value on the hanger. Ask students to compare the two strategies and discuss how they are alike and how they are different.

3.4: Weight of the Circle (Cool-down, 5 minutes)

Setup: None.

Student task statement

Here is a balanced hanger.



1. Write an equation representing this hanger.
2. Find the weight of one circle. Show or explain how you found it.

Possible responses

1. $4w = 25$
2. $\frac{25}{4}$ or $6\frac{1}{4}$

Unit 6, Lesson 4: Practice Solving Equations and Representing Situations with Equations

Lesson Goals

- Represent addition and multiplication situations with equations in different forms.
- Fluently solve equations of the form $x + p = q$ and $px = q$.

Required Materials

4.1: Number Talk: Subtracting From Five (5 minutes)

Setup: Display one problem at a time. Allow 30 seconds of quiet think time per problem, followed by a whole-class discussion.

Student task statement

Find the value of each expression mentally.

$5 - 2$

$5 - 2.1$

$5 - 2.17$

$5 - 2\frac{7}{8}$

Possible responses

- 3
- 2.9
- 2.83
- $2\frac{1}{8}$

4.2: Row Game: Solving Equations Practice (15 minutes)

Setup:

Clarify the meaning of "solving an equation." Students in groups of 2. Ensure students understand how the row game works. 10 minutes of partner work, followed by a whole class-discussion.

Student task statement

Solve the equations in one column. Your partner will work on the other column.

Check in with your partner after you finish each row. Your answers in each row should be the same. If your answers aren't the same, work together to find the error and correct it.

column A	column B
$18 = 2x$	$36 = 4x$
$17 = x + 9$	$13 = x + 5$
$8x = 56$	$3x = 21$
$21 = \frac{1}{4}x$	$28 = \frac{1}{3}x$
$6x = 45$	$8x = 60$
$x + 4\frac{5}{6} = 9$	$x + 3\frac{5}{6} = 8$
$\frac{5}{7}x = 55$	$\frac{3}{7}x = 33$
$\frac{1}{5} = 6x$	$\frac{1}{3} = 10x$
$2.17 + x = 5$	$6.17 + x = 9$
$\frac{20}{3} = \frac{10}{9}x$	$\frac{14}{5} = \frac{7}{15}x$
$14.88 + x = 17.05$	$3.91 + x = 6.08$
$3\frac{3}{4}x = 1\frac{1}{4}$	$\frac{7}{5}x = \frac{7}{15}$

Possible responses

1. $x = 9$
2. $x = 8$
3. $x = 7$
4. $x = 84$
5. $x = 7\frac{1}{2}$
6. $x = 4\frac{1}{6}$
7. $x = 77$
8. $x = \frac{1}{30}$
9. $x = 2.83$
10. $x = 6$
11. $x = 2.17$
12. $x = \frac{1}{3}$

4.3: Choosing Equations to Match Situations (15 minutes)

Setup: 10 minutes of quiet work time, followed a by whole-class discussion.

Student task statement

- Circle **all** of the equations that describe each situation. If you get stuck, draw a diagram.
- Find the solution for each situation.

Possible responses

1. $26 - x = 8$
   $x + 8 = 26$
   $26 - 8 = x$
   $x = 18$

2. $\frac{y}{8} = 14$
   $\frac{1}{8}y = 14$
   $y = 14 \cdot 8$
   $y = 112$

3. $223 = 409 - z$
   $409 - 223 = z$
   $409 = 223 + z$
   $z = 186$

4. $3w = 27$
   $w = \frac{1}{3} \cdot 27$
   $w = 27 \div 3$
   $w = 9$

Anticipated misconceptions

Students who continue to focus on key words misidentify the relationship in each situation. Encourage students

1. Clare has 8 fewer books than Mai. If Mai has 26 books, how many books does Clare have?

   - $26 - x = 8$
   - $x = 26 + 8$
   - $x + 8 = 26$
   - $26 - 8 = x$

   $x =$ ________

2. A coach formed teams of 8 from all the players in a soccer league. There are 14 teams. How many players are in the league?

   - $y = 14 \div 8$
   - $\frac{y}{8} = 14$
   - $\frac{1}{8}y = 14$
   - $y = 14 \cdot 8$

   $y =$ ________

3. Kiran scored 223 more points in a computer game than Tyler. If Kiran scored 409 points, how many points did Tyler score?

   - $223 = 409 - z$
   - $409 - 223 = z$
   - $409 + 223 = z$
   - $409 = 223 + z$

   $z =$ ________

4. Mai ran 27 miles last week, which was three times as far as Jada ran. How far did Jada run?

   - $3w = 27$
   - $w = \frac{1}{3} \cdot 27$
   - $w = 27 \div 3$
   - $w = 3 \cdot 27$

   $w =$ ________

to express the relationships in their own words and draw diagrams comparing the given quantities. For example, in the situation with Clare and Mai, they can draw a long rectangle representing Mai's books subdivided into two pieces. Filling in the information given in the story will help clear up the relationships; Clare's rectangle is labeled x, and she has 8 fewer books than Mai, so Mai's rectangle is labeled $x + 8$ and also 26. Alternatively, they can show that the piece labeled x must equal $26 - 8$.

Are you ready for more?

Mai's mother was 28 when Mai was born. Mai is now 12 years old. In how many years will Mai's mother be twice Mai's age? How old will they be then?

Possible Responses

16 years; Mai will be 28 and her mother will be 56.

Lesson Synthesis (5 minutes)

Encourage students to reflect on the learning of the past four lessons. How do diagrams and equations help us understand relationships between quantities? What diagrams have you seen and when were they helpful or not helpful? What does it mean to solve an equation and why is it useful?

4.4: More Storytime (Cool-down, 5 minutes)

Setup: None.

Student task statement

1. Write a story to match the equation $x + 2\frac{1}{2} = 10$.
2. Explain what x represents in your story.
3. Solve the equation. Explain or show your reasoning.

Possible responses

1. Answers vary.
2. Answers vary.
3. $x = 7\frac{1}{2}$

Unit 6, Lesson 5: A New Way to Interpret a over b

Lesson Goals

- Apply understanding of a fraction as a division to solve equations of the form $px = q$.
- Given an equation of the form $x + p = q$ or $px = q$, write a story that the equation might represent, explain what quantity the variable represents, and solve the equation.

Required Materials

5.1: Recalling Ways of Solving (5 minutes)

Setup: 1-2 minutes quiet think time, followed by a whole-class discussion.

Student task statement

Solve each equation. Be prepared to explain your reasoning.

1. $0.07 = 10m$
2. $10.1 = t + 7.2$

Possible responses

1. $0.007 = m$
2. $2.9 = t$

5.2: Interpreting $\frac{a}{b}$ (15 minutes)

Setup:

Groups of 2. 5–10 minutes of quiet work time and time to share their responses with a partner, followed by a whole-class discussion.

Student task statement

Solve each equation.

1. $35 = 7x$
2. $35 = 11x$
3. $7x = 7.7$
4. $0.3x = 2.1$
5. $\frac{2}{5} = \frac{1}{2}x$

Possible responses

1. 5
2. $\frac{35}{11}$
3. 1.1
4. 7
5. $\frac{4}{5}$

Anticipated misconceptions

Monitor for students who want to turn $\frac{35}{11}$ into a decimal, and reassure them that $\frac{35}{11}$ is a number.

Are you ready for more?

Solve the equation. Try to find some shortcuts.

$$\frac{1}{6} \cdot \frac{3}{20} \cdot \frac{5}{42} \cdot \frac{7}{72} \cdot x = \frac{1}{384}$$

Possible Responses

$x = 9$. Solution methods vary. One way is to factor each denominator and notice that there are many numbers that occur in both numerator and denominator.

5.3: Storytime Again (15 minutes)

Setup: Keep students in the same groups. 5–10 minutes to work with their partner, followed by a whole-class discussion.

Student task statement

Take turns with your partner telling a story that might be represented by each equation. Then, for each equation, choose one story, state what quantity x describes, and solve the equation. If you get stuck, draw a diagram.

1. $0.7 + x = 12$
2. $\frac{1}{4}x = \frac{3}{2}$

Possible responses

Stories vary. Solutions to equations:

1. $x = 11.3$
2. $x = 6$

Anticipated misconceptions

For students with limited fraction and decimal understanding, coming up with a reasonable story where the numbers are not whole can be daunting. You might suggest that students imagine stories with similar structures that involve whole numbers, and then tweak the stories toward using the numbers given in the problems. Remind them that using fractions and decimals has to make sense in the situations, and encourage them to think about what kinds of situations those might be (measurement situations will usually work while those that involve counting discrete objects won't.)

Lesson Synthesis (5 minutes)

Ask partners to choose two numbers for p and q in an equation of the form $px = q$ where both are fractions or decimals, write the equation, and then solve it. Ensure students see that when solving equations of the form $px = q$ where p and q may not be whole numbers, we can interpret the solution $x = q \div p$ as the fraction $\frac{q}{p}$.

5.4: Choosing Solutions (Cool-down, 5 minutes)

Setup: None.

Student task statement

Select **all** the expressions that are solutions to $5 = \frac{2}{3}x$.

A. $5 \cdot \frac{2}{3}$

B. $\frac{5}{\frac{2}{3}}$

C. $5 \div \frac{2}{3}$

D. $\frac{15}{2}$

E. $\frac{10}{3}$

Possible responses

B, C, D

Unit 6, Lesson 6: Write Expressions Where Letters Stand for Numbers

Lesson Goals

- Write an expression with a variable to describe a situation with an unknown amount.
- Use an expression to write an equation about a situation and solve the equation to find an unknown quantity.

Required Materials

6.1: Algebra Talk: When x is 6 (5 minutes)

Setup: Display one problem at a time. Allow 30 seconds of quiet think time per problem, followed by a whole-class discussion.

Student task statement	Possible responses
If x is 6, what is:	
• $x+4$	• 10
• $7-x$	• 1
• x^2	• 36
• $\frac{1}{3}x$	• 2