ACE

Assignment Guide for Problem 3.2

Applications: 5–20 | Connections: 31–36
Extensions: 51–53

Answers to Problem 3.2

A. **1.** $36 = 2 \times 2 \times 3 \times 3$; $36 = 2^2 \times 3^2$

2. A student might choose the factor pair (4, 9), write out the factorization of 36 as $36 = 2 \times 2 \times 3 \times 3$, and circle 2×2 and 3×3.

$36 = (2 \times 2) \times (3 \times 3)$

$36 = 4 \times 9$

Make sure students see that the prime factorization of 36 is composed of the prime factorization of 4 and the prime factorization of 9.

3. To find the first number in a factor pair, first choose some of the prime numbers in the factorization. Multiply these primes together. To find the second factor in the pair, multiply the remaining prime factors. For example, a student might circle $2 \times 3 \times 3$ and 2 to find the factor pair (18, 2).

4. (2, 18): $36 = 2 \times 2 \times 3 \times 3$
$36 = 2 \times (2 \times 3 \times 3)$
$36 = 2 \times 18$

(3, 12): $36 = 2 \times 2 \times 3 \times 3$
$36 = (2 \times 2 \times 3) \times 3$
$36 = 12 \times 3$

(4, 9): $36 = 2 \times 2 \times 3 \times 3$
$36 = (2 \times 2) \times (3 \times 3)$
$36 = 4 \times 9$

(6, 6): $36 = 2 \times 2 \times 3 \times 3$
$36 = 2 \times 3 \times 2 \times 3$
$36 = (2 \times 3) \times (2 \times 3)$
$36 = 6 \times 6$

5. One possible multiple of 36 is 72. The prime factorization of 72 contains the prime factorization of 36.

B. **1.** **a.** $10 = 2 \times 5$

b. $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$

c. $1{,}000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^3 \times 5^3$

d. $10{,}000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5 = 2^4 \times 5^4$

2. 270,000 is the same as $27 \times 10{,}000$. Therefore, $270{,}000 = 3 \times 3 \times 3 \times 10^4$. By using the observation from Question B, part (1), we can write the prime factorization of 270,000 as $3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5$ or $3^3 \times 2^4 \times 5^4$.

C. **1.** $2^4 \times 3^2 \times 5 = 720$

2. Yes; $2^2 \times 3 = 12$ is a factor of 720, because $12 \times 60 = 720$, or because the factorization $2^2 \times 3$ is included in the factorization $2^4 \times 3^2 \times 5$.

3. Yes; $2^5 \times 3^2 \times 5 = 1{,}440$ is twice 720 (i.e., $2 \times 2^4 \times 3^2 \times 5$).

D. Since $6 = 2 \times 3$ and $12 = 2 \times 2 \times 3$, $2 \times 5^2 \times 6 \times 11 \times 12$ becomes $2^4 \times 3^2 \times 5^2 \times 11$.

Since $25 = 5 \times 5$ and $44 = 2 \times 2 \times 11$, $3 \times 11 \times 25 \times 44$ becomes $2^2 \times 3 \times 5^2 \times 11^2$.

Since $12^2 = (2 \times 2 \times 3)^2$ and $55 = 5 \times 11$, $5 \times 12^2 \times 55$ becomes $2^4 \times 3^2 \times 5^2 \times 11$.

Since $10 = 2 \times 5$, $22 = 2 \times 11$, and $33 = 3 \times 11$, $5 \times 10 \times 22 \times 33$ becomes $2^2 \times 3 \times 5^2 \times 11^2$.

Therefore, $3 \times 11 \times 25 \times 44$ and $5 \times 10 \times 22 \times 33$ are equivalent, and $2 \times 5^2 \times 6 \times 11 \times 12$ and $5 \times 12^2 \times 55$ are equivalent. **Note:** Look for other ways that students might answer this question.

E. There is only one (unique) set of prime factors for any number. For example, the prime factors of 330 are 2, 3, 5 and 11, and the prime factorization of 330 is $2 \times 3 \times 5 \times 11$. There is no other possible set of prime numbers that can be multiplied to make 330. The prime numbers cannot be broken down further into a product of primes. 3 can be written as 1×3, but 1 is not a prime.

- *Now let's describe some strategies for finding LCM of 72 and 120. How can you check to make sure 360 is a common multiple of 72 and 120?*
- *Could a lesser number be a common multiple of 72 and 120?*
- *List five more common multiples of 72 and 120 and tell how they relate to the LCM.*
- *Is 24 a factor of 60?*
- *How can you use the prime factorizations of 24 and 60 to see that 24 is not a factor of 60?*
- *How do you know that 14 is not a factor of either 24 or 60?*

Be sure to summarize Questions B and C. You can use Question D as a final check for understanding.

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Assignment Guide for Problem 3.3

Applications: 21–27 | Connections: 37–42
Extensions: 54–55

Answers to Problem 3.3

A. **1.** The GCF is 24. $72 = 2 \times 2 \times 2 \times 3 \times 3$; $120 = 2 \times 2 \times 2 \times 3 \times 5$; The longest factor string they have in common is $2 \times 2 \times 2 \times 3$. Therefore, the GCF is $2 \times 2 \times 2 \times 3 = 24$.

2. The LCM is 360. The factor string that 72 and 120 have in common is $2 \times 2 \times 2 \times 3$. The prime factorization of 72 includes one additional 3. The prime factorization of 120 includes a 5. Therefore, the factor string that includes the prime factorization of both numbers, without common duplicate factors, is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$.

B. **1.** Some possible answers: 15 and 8, 7 and 20, 34 and 35.

2. If two numbers have no prime factors in common, their only common factor is 1.

C. **1.** Possible answers: 5 and 4; 8 and 63. If the prime factorizations have no factors in common, the LCM of the two numbers will be the product of the numbers.
Note: You may want to point out that the LCM of two distinct prime numbers is always the product of the two numbers. Of course, it is not always true that the LCM of two composite numbers is less than the product. You might push students to make generalizations. Students should be able to find many examples.

2. Possible answers: 8 and 40; 10 and 15. The LCM of two numbers will be less than the product of the numbers whenever the numbers have factors in common.

3. The LCM of two numbers will be less than the product of the numbers whenever the numbers have factors, or primes, in common. The LCM is equal to the product when the numbers have no common factors.

D. **1.** Possible answers: 2 and 45; 9 and 10. Their factorizations have no factors in common and their product is 90. Additional possible answers: 6 and 45; 30 and 45. The product of the union of their prime factorizations is equal to 90.

2. Possible answers: 90 and 180; 180 and 270. Their factorizations have the common factor $90 = 2 \times 5 \times 3^2$ and share no additional prime factors. In general, any two numbers $90 \times p$ or $90 \times q$ where p and q are relatively prime numbers.

- *How can you determine the last locker touched by both students?*
- *A student touches two different lockers. What do you know about the student number and the two lockers?*
- *Who will be the last student to touch both of those two lockers?*
- *If I give you two locker numbers, what strategy would you use to determine which students touched both of them?*

As you discuss the Problem, ask students to translate the questions from the language of lockers to the language of mathematics and vice versa, from mathematics to lockers.

As you come back to the Focus Question, be sure to ask students to provide examples.

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Assignment Guide for Problem 3.4

Applications: 28–30 | Connections: 43–50
Extensions: 56

Answers to Problem 3.4

A. **1.** Students may see many different patterns. For example, students might notice that prime-numbered lockers are closed because they have only two factors.

2. Lockers that have square numbers remain open. They have an odd number of factors. Therefore, the final action will be to open the locker. For example, 16 has five factors: 1, 2, 4, 8, 16. The sequence of the door would be O, C, O, C, O. The square numbers less than 30 are 1, 4, 9, 16, and 25.

3. The lockers that are open, that is, the square numbers less than 1,000, are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961.

B. **1.** The lockers with prime numbers, because they have only two factors. The first few of these are 2, 3, 5, 7, 11, 13, 17, 19, 23, and so on.

2. The lockers with the squares of prime numbers, which are 4, 9, 25, 49, 121, 169, 289, 361, 529, 841, and 961.

3. The lockers with the numbers that are products of two different primes or numbers whose prime factorization is the cube of a prime number. Examples are 6 (2×3), 10 (2×5), 15 (3×5), 8 ($2 \times 2 \times 2$), 27 ($3 \times 3 \times 3$), and so on.

4. By finding the number of factors of the given locker number.

C. **1.** Locker 24, because 24 is the LCM of 6 and 8.

2. Locker 60, because 60 is the LCM of 12 and 30.

3. Locker 91, because 91 is the LCM of 7 and 13.

4. Locker 600, because 600 is the LCM of 100 and 120.

5. Since each student touches lockers that are multiples of the student's number, you can find the first locker they both touch by finding the LCM of the students' numbers. The last locker touched by both students would be the greatest common multiple of the students' numbers that is less than 1,000.

D. **1.** Students 1, 2, 3, 4, 6, and 12, because these are the common factors of 24 and 36.

2. Students 1, 2, 4, 5, 10, and 20, because these are the common factors of 100 and 120.

3. Students 1, 3, 7, and 21, because these are the common factors of 42 and 273.

4. You can find the common factors of the two numbers.

ACE Answers Investigation 3

Applications

1. The path is $7 \times 5 \times 2 \times 3 \times 4$.
2. The path is $3 \times 4 \times 5 \times 6$.
3. Mazes will vary.
4. He can give each child 3 cookies, and he will have 12 left for himself.
5. $2 \times 2 \times 3 \times 3$
6. $2 \times 2 \times 3 \times 3 \times 5$
7. $3 \times 5 \times 5 \times 7$
8. $3 \times 5 \times 11$
9. 293
10. $2 \times 2 \times 2 \times 5 \times 19$
11. $2 \times 2 \times 2 \times 3 \times 3 \times 3$
12. $3 \times 7 \times 11$
13. $2 \times 2 \times 2 \times 3 \times 13$
14. $36 = 2^2 \times 3^2$, $180 = 2^2 \times 3^2 \times 5$, $525 = 3 \times 5^2 \times 7$, $165 = 3 \times 5 \times 11$, $293 = 293$, $760 = 2^3 \times 5 \times 19$, $216 = 2^3 \times 3^3$, $231 = 3 \times 7 \times 11$, and $312 = 2^3 \times 3 \times 13$
15. D
16. Jamahl is correct. Possible answer: Consider 216. It has six prime factors: $2 \times 2 \times 2 \times 3 \times 3 \times 3$. Then consider 231. It has three prime factors: $3 \times 7 \times 11$. 231 is greater than 216, even though it has fewer prime factors.
17. 10, 20, 30, 40, 50, 60, 70, 80, and 90. They are all multiples of 10.
18. H; $2 \times 3 \times 5$
19. $2 \times 3 \times 5 = 30$, $2 \times 3 \times 7 = 42$, $2 \times 3 \times 11 = 66$, $2 \times 3 \times 13 = 78$, $2 \times 5 \times 7 = 70$
20. a. $8 \times 8 = 64$, so Mr. and Mrs. Fisk have 64 grandchildren.
    b. $64 \times 8 = 512$, so they have 512 great-grandchildren.
    c. Expressions may vary. Sample: $8^3 = 512$.
21. GCF = 9, LCM = 180
22. GCF = 15, LCM = 150
23. GCF = 26, LCM = 312
24. GCF = 15, LCM = 60
25. GCF = 1, LCM = 1,440
26. GCF = 1, LCM = 444
27. a. $N = 45$, 90, or 180. Any other factor of 180 has a common multiple with 12 that is less than 180.
    b. $M = 14$ or any multiple of 14 that doesn't have a factor of 3 or 5.
28. 16, 81, and 625 (the numbers that are a prime number to the 4th power)
29. a. Locker 15
    b. Locker 60
    c. Locker 504
    d. Locker 630
30. a. Only Student 1 touched both lockers because the locker numbers are relatively prime.
    b. Students 1, 2, 5, 7, 10, 14, 35, and 70 touched both lockers 140 and 210. These are the common factors.
    c. Students 1, 3, 5, 11, 15, 33, 55, and 165 touched both lockers 165 and 330. These are the common factors.
    d. Students 1, 2, 7, 14, 49, and 98 touched both lockers 196 and 294. These are the common factors.

Connections

31. Rosa is correct because the number 1 is not a prime number. Tyee is correct that his string is a longer string of factors, but it is not a string of prime factors for 30.

32. Mathematicians have determined that it is important for a number to be able to be identified by its longest string of factors. If the number 1 were prime, the prime factorization for a number would have to include 1 and could include 1 as a factor any number of times. So a prime factorization would not be the same as the longest string possible without using 1.

33. **a.** $100 = 2^2 \times 5^2$, which has nine factors (1, 2, 4, 5, 10, 20, 25, 50, 100).

 b. 101 is prime and has only two factors (1, 101).

 c. $102 = 2 \times 3 \times 17$, which has eight factors (1, 2, 3, 6, 17, 24, 51, 102).

 d. 103 is prime and has only two factors (1, 103).

 e. Answers may vary. Students may notice that for four consecutive numbers, 2 is a factor of two of the numbers; 3 is a factor of at least one of the numbers; 4 must be a factor of one of the numbers; larger numbers do not necessarily have more factors.

34. These numbers are all multiples of 6.

35. These numbers are all multiples of 15.

36. 1, 2, 3, 5, 15, and 30 are also common factors.

37. **a.** 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, and 99

 b. 21, 42, 63, and 84

 c. 63

 d. 126

38. Factor to find that $184 = 8 \times 23$ and $207 = 9 \times 23$. 23 is the only common facto other than 1. If they only earned \$1 a day, they would have to work longer than one month to earn \$184 or more. Therefore, Tomas worked 8 days at \$23 per day and Sharina worked 9 days at \$23 per day.

39. Answers will vary. You might want to have students post some of their stories around the room. **Note:** The prime factorization o 648 is $648 = 2^3 \times 3^4$.

40. 53

41. Yes. Numbers that end in 0 are multiples o 10. Numbers that are multiples of 10 have both 2 and 5 as factors.

42. 70. Since the number is a multiple of 2 and 7 (Clue 1), which are relatively prime, the number must be a multiple of 14. The multiples of 14 between 50 and 100 (Clue 2) are $56 = 2 \times 2 \times 2 \times 7$, $70 = 2 \times 5 \times 7$, $84 = 2 \times 2 \times 3 \times 7$, and $98 = 2 \times 7 \times 7$. Of these numbers, only 70 is the product of three different prime numbers (Clue 3).

43. The factors of 32 are 1, 2, 4, 8, 16, and 32 (Clue 3). Of these numbers, only 1, 16 and 32 have digits that sum to odd numbers (Clue 4). 1 and 16 are square numbers (Clue 1). Of these two numbers, only 16 has 2 in its prime factorization (Clue 2). Th number is 16.

44. A number that is a multiple of 3 (Clue 2) and of 5 (Clue 1) must be a multiple of 15. The multiples of 15 that are less than 50 are 15, 30, and 45. Only 30 has 8 factors (Clue 3).

45. Multiples of 5 that don't end in 5 are multiples of 10 (Clue 1). The factor string is three numbers long (Clue 2), and two of these are 2 and 5. Since two of the numbers in the factor string are the same (Clue 3), the number is $2 \times 2 \times 5 = 20$ or $2 \times 5 \times 5 = 50$. The number is greater than the seventh prime, 49, so 50 is the number.

46. Answers will vary. You may want to post some of the best student responses for a Problem of the Week.

47. Only Locker 42 meets all three criteria. A good way to approach the problem is to list all the even numbers 50 or less and cross out those that fail to meet each of the other criteria. A student might also start by listing only the numbers divisible by 7 and then apply the other two criteria to those numbers.

48. a. 2, 4, 8, 16, 32, 64, 128, 256, 512

b. 1,024

49. This is the only pair of primes that are consecutive numbers. We know that this is the only such pair because 2 is the only even prime number.

50. There are (infinitely) more odd prime numbers. The only even prime is 2.

Extensions

51. a. Possible answers:
$1994 = 2 \times 997$, $1995 = 3 \times 5 \times 7 \times 19$,
$1996 = 2 \times 2 \times 499$, $1997 = 1997$,
$1998 = 2 \times 3 \times 3 \times 3 \times 37$,
$1999 = 1999$,
$2000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5$,
$2001 = 3 \times 23 \times 29$,
$2002 = 2 \times 7 \times 11 \times 13$, $2003 = 2003$,
$2004 = 2 \times 2 \times 3 \times 167$,
$2005 = 5 \times 401$, $2006 = 2 \times 17 \times 59$,
$2007 = 3 \times 3 \times 223$,
$2008 = 2 \times 2 \times 2 \times 251$,
$2009 = 7 \times 7 \times 41$,
$2010 = 2 \times 3 \times 5 \times 67$

b. Answers will vary. For example, 1996 is not square; it is a multiple of 4 and of 998. It is not prime, and it is even.

52. a. 52 weeks, with 1 extra day if it is not a leap year and 2 extra days if it is a leap year.

b. January 8, 15, and 22

c. Monday

d. Tuesday

e. Your birthday will fall one day later in the week each year, except when leap day (February 29) falls between your birthdays. In that case, your birthday will be two days later in the week.
If your birthday is February 29, your birthday will be five days later in the week each time it occurs.

53. Answers will vary. Sample: If a number is the least common multiple of several prime numbers, its prime factorization will include only those primes, and no others.

54. The common multiples of 2, 3, 4, 5, and 6 are 60, 120, 180, ... If we add the clue that the box contains fewer than 100 books, the only answer would be 60.

55. a. Barry is correct. The $\text{LCM}(a,b) = a \times b \div \text{GCF}(a,b)$. When you multiply a and b together, you will get each of the overlapping prime factors twice. Therefore, if you divide by the GCF, you will get only one set of overlapping factors, along with the factors that do not overlap.

b. The product of the LCM and the GCF is equal to the product of the numbers.

56. Yes. The concept behind David's prediction is that the prime factorization of a number determines all of its factors. If a number is divisible by 2, for instance, then 2 will be a factor of many of the number's factors. In fact, each factor of a number is built up of one or more of the number's other factors.

Consider 40. The prime factors of 40 are 2 and 5; the prime factorization is $2 \times 2 \times 2 \times 5$, or $2^3 \times 5^1$.

Consider each of 40's factors in terms of its own prime factorization:

$1 = 2^0 \times 5^0$

$2 = 2^1 \times 5^0$

$4 = 2^2 \times 5^0$

$8 = 2^3 \times 5^0$

$5 = 2^0 \times 5^1$

$10 = 2^1 \times 5^1$

$20 = 2^2 \times 5^1$

$40 = 2^3 \times 5^1$

If we look at all of those numbers in terms of their prime factorizations, we can see every possible arrangement of 2 to the 0, 1, 2, or 3 power with 5 to the 0 or 1 power (i.e., it's possible to raise the prime factor to the zero power).